

Uniqueness of transverse solutions for reaction-diffusion equations with spatially distributed hysteresis

Pavel Gurevich ^{*}, Sergey Tikhomirov [†]

March 4, 2013

Abstract

The paper deals with reaction-diffusion equations involving a hysteretic discontinuity in the source term, which is defined at each spatial point. Such problems describe biological processes and chemical reactions in which diffusive and nondiffusive substances interact according to hysteresis law. Under the assumption that the initial data are spatially transverse, we prove a theorem on the uniqueness of solutions. The theorem covers the case of non-Lipschitz hysteresis branches arising in the theory of slow-fast systems.

Key words. spatially distributed hysteresis, reaction-diffusion equation, uniqueness of solution.

AMS subject classification. 35K57, 35K45, 47J40

1 Introduction

We consider reaction-diffusion equations with right-hand sides involving a discontinuous hysteresis defined at each spatial point. Such problems describe biological processes and chemical reactions in which diffusive and nondiffusive substances interact according to hysteresis law. As a result, various spatial and spatio-temporal patterns may appear (see, e.g., [5, 6]).

First rigorous results about the existence of solutions of parabolic equations with hysteresis in the source term have been obtained in [1, 9, 16] for multi-valued hysteresis. Formal asymptotic expansions of solutions were recently obtained for some special case in [8]. However, the uniqueness of solutions and their continuous dependence on initial data as well as a thorough analysis of pattern formation remained open questions.

In [3], a new approach was suggested. It allowed us to find a broad class of initial data (transverse functions, see Sec. 2) for which a solution exists and, if unique, continuously depends on initial data. The approach is based on tracking the so-called free boundary which defines the hysteresis topology. The hysteresis topology is, in the simplest case, related to the structure of subdomains in space where hysteresis takes the same value. The main advantage of this approach is that, tracking the hysteretic free boundary, one gets necessary information on the precise form of emerging spatio-temporal patterns.

The transversality assumption roughly speaking means that the initial function has a nonvanishing derivative on the boundary between the above-mentioned subdomains at the initial moment (see Condition 2.2). Under this assumption, we prove in the present paper that the solution for the reaction-diffusion equation with discontinuous spatially distributed hysteresis is unique for some positive time. Combining this result with the results in [3], we obtain that the solution can be uniquely continued as long as it remains transverse.

We consider a one-dimensional domain and, for the clarity of exposition, we concentrate on a scalar reaction-diffusion equation. The paper is organized as follows. In Sec. 2, we define functional spaces, introduce spatially distributed hysteresis and set the prototype problem. In the end of Sec. 2, we formulate the main result of the paper: Theorem 2.1 on the uniqueness of transverse solutions.

^{*}Free University of Berlin, Peoples' Friendship University of Russia, email: gurevichp@gmail.com

[†]Free University of Berlin, email: sergey.tikhomirov@gmail.com

Section 3 is devoted to the proof of Theorem 2.1.

Interestingly, the uniqueness of solutions holds for some classes non-Lipschitz hysteresis branches, too. In particular, we prove the uniqueness for hysteresis branches arising in bistable slow-fast reaction-diffusion systems (see e.g., [13] and the references therein). In Appendix A, we briefly describe such systems and show that the arising hysteresis branches satisfy assumptions of our main theorem.

2 Setting of the problem

2.1 Functional spaces

We denote by $L_q = L_q(0, 1)$, $q > 1$, the standard Lebesgue space and by $W_q^l = W_q^l(0, 1)$ with natural l the standard Sobolev space. For a noninteger $l > 0$, denote by $W_q^l = W_q^l(0, 1)$ the Sobolev space with the norm

$$\|v\|_{W_q^l} = \|v\|_{W_q^{[l]}} + \left(\int_0^1 dx \int_0^1 \frac{|v^{([l])}(x) - v^{([l])}(y)|^q}{|x - y|^{1+q(l-[l])}} dy \right)^{1/q},$$

where $[l]$ is the integer part of l .

Let $Q_T = (0, 1) \times (0, T)$. We introduce the Hölder space $C^\gamma(\overline{Q}_T)$, $0 < \gamma < 1$, and the anisotropic Sobolev space $W_q^{2,1}(Q_T)$ with the norm

$$\|u\|_{W_q^{2,1}(Q_T)} = \left(\int_0^T \|u(\cdot, t)\|_{W_q^2}^q dt + \int_0^T \|u_t(\cdot, t)\|_{L_q}^q dt \right)^{1/q}.$$

Throughout the paper, we fix q and γ such that

$$q > 3, \quad 0 < \gamma < 1 - 3/q.$$

Then $u, u_x \in C^\gamma(\overline{Q}_T)$ whenever $u \in W_q^{2,1}(Q_T)$ (see Lemma 3.3 in [12, Chap. 2]).

In what follows, we will consider solutions of parabolic problems in the space $W_q^{2,1}(Q_T)$. To define the space of initial data, we will use the fact that if $u \in W_q^{2,1}(Q_T)$, then the trace $u|_{t=t_0}$ is well defined and belongs to $W_q^{2-2/q}$ for all $t_0 \in [0, T]$ (see Lemma 2.4 in [12, Chap. 2]). Moreover, one can define the space $W_{q,N}^{2-2/q}$ as the subspace of functions from $W_q^{2-2/q}$ with the zero Neumann boundary conditions.

2.2 Hysteresis

In this section, we introduce a hysteresis operator defined for functions of time variable t . Then we extend the definition to a spatially distributed hysteresis acting on a space of functions of time variable t and space variable x .

We fix two numbers α and β such that $\alpha < \beta$. The numbers α and β will play a role of thresholds for the hysteresis operator. Next, we introduce continuous functions (*hysteresis branches*)

$$H_1 : (-\infty, \beta] \mapsto \mathbb{R}, \quad H_2 : [\alpha, \infty) \mapsto \mathbb{R}.$$

We assume throughout that the following condition holds.

Condition 2.1. *There is a number $\sigma \in [0, 1)$ such that, for any $U > 0$, there exists $M = M(U) > 0$ with the properties*

$$|H_1(u) - H_1(\hat{u})| \leq \frac{M}{(\beta - u)^\sigma + (\beta - \hat{u})^\sigma} |u - \hat{u}|, \quad \forall u, \hat{u} \in [-U, \beta], \quad (2.1)$$

$$|H_2(u) - H_2(\hat{u})| \leq \frac{M}{(u - \alpha)^\sigma + (\hat{u} - \alpha)^\sigma} |u - \hat{u}|, \quad \forall u, \hat{u} \in (\alpha, U]. \quad (2.2)$$

Remark 2.1. 1. Any locally Lipschitz continuous functions $H_1(u)$ and $H_2(u)$ satisfy Condition 2.1. Moreover, this condition covers the important case where the hysteresis branches $H_1(u)$ and $H_2(u)$ are the stable parts of the curve $g(u, v) = 0$ in the slow-fast system (A.1). They are not Lipschitz continuous near the points $u = \beta$ and α , respectively. But they still satisfy Condition 2.1 (see Appendix A for details).

2. On the other hand, any $H_1(u)$ and $H_2(u)$ satisfying Condition 2.1 are locally Hölder continuous with exponent $1-\sigma$ on $(-\infty, \beta]$ and $[\alpha, \infty)$, respectively. Furthermore, $H_1(u)$ and $H_2(u)$ are locally Lipschitz on the open intervals $(-\infty, \beta)$ and (α, ∞) , respectively. Therefore, they satisfy the assumptions in [3], where existence of solutions and their continuous dependence on initial data were proved.

We fix $T > 0$ and denote by $C_r[0, T)$ the linear space of functions which are continuous on the right in $[0, T)$. For any $\zeta_0 \in \{1, 2\}$ (*initial configuration of hysteresis*) and $g \in C[0, T]$ (*input*), we introduce the *configuration function*

$$\zeta : \{1, 2\} \times C[0, T] \rightarrow C_r[0, T), \quad \zeta(t) = \zeta(\zeta_0, g)(t)$$

as follows. Let $X_t = \{t' \in (0, t] : g(t') = \alpha \text{ or } \beta\}$. Then

$$\zeta(0) = \begin{cases} 1 & \text{if } g(0) \leq \alpha, \\ 2 & \text{if } g(0) \geq \beta, \\ \zeta_0 & \text{if } g(0) \in (\alpha, \beta) \end{cases}$$

and for $t \in (0, T]$

$$\zeta(t) = \begin{cases} \zeta(0) & \text{if } X_t = \emptyset, \\ 1 & \text{if } X_t \neq \emptyset \text{ and } g(\max X_t) = \alpha, \\ 2 & \text{if } X_t \neq \emptyset \text{ and } g(\max X_t) = \beta. \end{cases}$$

Now we introduce the *hysteresis operator* (cf. [11, 15])

$$\mathcal{H} : \{1, 2\} \times C[0, T] \rightarrow C_r[0, T)$$

by the following rule. For any initial configuration $\zeta_0 \in \{1, 2\}$ and input $g \in C[0, T]$, the function $\mathcal{H}(\zeta_0, g) : [0, T] \rightarrow \mathbb{R}$ (*output*) is given by

$$\mathcal{H}(\zeta_0, g)(t) = H_{\zeta(t)}(g(t)),$$

where $\zeta(t)$ is the configuration function defined above (see Fig. 2.1). Note that this hysteresis operator is typically not continuous.

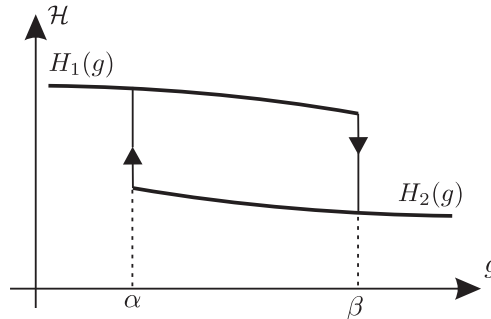


Figure 2.1: The hysteresis operator \mathcal{H}

Now we introduce a *spatially distributed hysteresis*. Assume that the initial configuration and the input function depend on spatial variable $x \in [0, 1]$. Denote them by $\xi_0(x)$ and $u(x, t)$, where

$$\xi_0 : [0, 1] \mapsto \{1, 2\}, \quad u : [0, 1] \times [0, T] \mapsto \mathbb{R}.$$

Let $u(x, \cdot) \in C[0, T]$. Denote $\varphi(x) = u(x, 0)$. We say that a function $\varphi(x)$ and a hysteresis configuration $\xi_0(x)$ are *consistent* if, for any $x \in [0, 1]$,

$$\xi_0(x) \in \begin{cases} \{1\} & \text{if } \varphi(x) \geq \beta, \\ \{2\} & \text{if } \varphi(x) \leq \alpha, \\ \{1, 2\} & \text{if } \varphi(x) \in (\alpha, \beta). \end{cases}$$

Then we can define the function

$$v(x, t) = \mathcal{H}(\xi_0(x), u(x, \cdot))(t), \quad (2.3)$$

which is called *spatially distributed hysteresis*.

2.3 Reaction-diffusion equations with hysteresis

The main object of this paper is the initial boundary-value problem for the reaction-diffusion equation

$$u_t = u_{xx} + v, \quad (x, t) \in Q_T, \quad (2.4)$$

$$u_x|_{x=0} = u_x|_{x=1} = 0, \quad (2.5)$$

$$u|_{t=0} = \varphi(x), \quad x \in (0, 1), \quad (2.6)$$

where $v = v(x, t)$ represents the spatially distributed hysteresis given by (2.3).

Remark 2.2. The more general equation

$$u_t = u_{xx} + f(u, v)$$

with locally Lipschitz continuous right-hand side f can be reduced to Eq. (2.4). Indeed, it suffices to replace the hysteresis branches $H_j(u)$ in the definition of hysteresis \mathcal{H} by $F_j(u) = f(u, H_j(u))$, $j = 1, 2$. One can check that the functions $F_j(u)$ also satisfy Condition 2.1.

The general assumption on the initial data $\varphi(x)$ under which the uniqueness result holds is that $\varphi(x)$ is transverse with respect to the initial configuration $\xi_0(x)$.

Definition 2.1. We say that a function $\varphi \in C^1[0, 1]$ is *transverse* (with respect to a spatial configuration $\xi_0(x)$) if it is consistent with $\xi_0(x)$ and the following holds:

1. if $\varphi(\bar{x}) = \alpha$ and $\varphi'(\bar{x}) = 0$ for some $\bar{x} \in [0, 1]$, then $\xi_0(\bar{x}) = 1$ in a neighborhood of \bar{x} ;
2. if $\varphi(\bar{x}) = \beta$ and $\varphi'(\bar{x}) = 0$ for some $\bar{x} \in [0, 1]$, then $\xi_0(\bar{x}) = 2$ in a neighborhood of \bar{x} .

For the clarity of exposition, we now restrict ourselves to the following prototype situation (see Fig. 2.2). Fix some $\bar{a} \in (0, 1)$.

Condition 2.2. 1. For $\bar{a} \in (0, 1)$, one has

$$\xi_0(x) = \begin{cases} 1, & x \leq \bar{a}, \\ 2, & x > \bar{a}. \end{cases}$$

2. The equation $\varphi(x) = \beta$ on the interval $[0, \bar{a}]$ has no roots.
3. The equation $\varphi(x) = \alpha$ on the interval $[\bar{a}, 1]$ has the unique root $x = \bar{a}$.
4. $\varphi'(\bar{a}) > 0$.

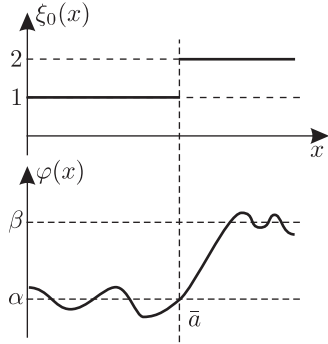


Figure 2.2: Initial data satisfying Condition 2.2

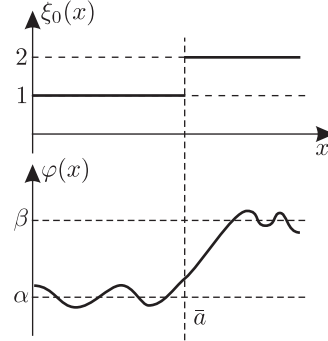


Figure 2.3: Initial data satisfying the assumptions of Remark 2.3

It follows from this condition that the hysteresis (2.3) at the initial moment is given by

$$v|_{t=0} = \begin{cases} H_1(\varphi(x)), & x \leq \bar{a}, \\ H_2(\varphi(x)), & x > \bar{a}. \end{cases}$$

We give a definition of a solution of problem (2.4)–(2.6), assuming that $\varphi \in W_{q,N}^{2-2/q}$.

Definition 2.2. A function $u(x, t)$ is called a *solution of problem (2.4)–(2.6) (in Q_T)* if $u \in W_q^{2,1}(Q_T)$, $v(x, t)$ is measurable, u and v satisfy equation (2.4) for a.e. $(x, t) \in Q_T$, and conditions (2.5) and (2.6) are satisfied in the sense of traces.

It follows from this definition that any solution $u(x, t)$ belongs to $C(\overline{Q_T})$. Therefore, the function $v(x, t)$ is well defined by (2.3) and belongs to $L_\infty(Q_T)$.

Remark 2.3. An initial function may also satisfy $\varphi(\bar{a}) > \alpha$. Such a function is also consistent with $\xi_0(x)$ from Condition 2.2, provided that the equation $\varphi(x) = \beta$ has no roots on the interval $[0, \bar{a}]$ and the equation $\varphi(x) = \alpha$ has no roots on the interval $[\bar{a}, 1]$ (see Fig. 2.3). However, this situation is much simpler. Indeed, one can show (see [3]) that, for any solution of problem (2.4)–(2.6) with such an initial function, the hysteresis $\mathcal{H}(\xi_0(x), u(x, \cdot))(t)$ does not depend on t on a sufficiently small time interval. Therefore, in this case, we leave details to the reader and rather concentrate on Condition 2.2.

2.4 Main result

In what follows, we always assume that Conditions 2.1 and 2.2 hold.

In [3], it was proved that any solution of problem (2.4)–(2.6) remains transverse on some time interval $[0, T]$ in the sense of the following definition.

Definition 2.3. A function $u \in C^{1,0}(\overline{Q_T})$ is *transverse on $[0, T]$* (with respect to a spatial configuration $\xi(x, t)$) if, for every fixed $t \in [0, T]$, the function $u(\cdot, t)$ is transverse with respect to the spatial configuration $\xi(\cdot, t)$.

Furthermore, problem (2.4)–(2.6) has at least one transverse solution on the above interval $[0, T]$ and any such solution can be extended to its maximal interval of existence on which it remains transverse.

In this paper, we show that there exists no more than one transverse solution of problem (2.4)–(2.6). We formulate the main result as follows.

Theorem 2.1. *Let Conditions 2.1 and 2.2 hold, and let $q > 3$. Assume that $u, \hat{u} \in W_q^{2,1}(Q_{T_0})$ are two transverse solutions of problem (2.4)–(2.6) in Q_{T_0} for some T_0 . Then $u = \hat{u}$.*

3 Proof of Theorem 2.1

Taking into account Remark 2.3, it suffices to prove the uniqueness on a sufficiently small time interval $[0, T]$, $T \leq T_0$.

We denote

$$\bar{\varphi} = \frac{\varphi_x(\bar{a})}{2}.$$

By Condition 2.2, $\bar{\varphi} > 0$.

Lemma 3.1. *Let $u \in W_q^{2,1}(Q_{T_0})$ be a solution of problem (2.4)–(2.6) for some $T_0 > 0$. Then there exist $T \in (0, T_0)$ and $\delta > 0$ such that the following hold on the interval $t \in [0, T]$:*

1. $u_x(x, t) \geq \bar{\varphi}$ for all $x \in [\bar{a} - \delta, \bar{a} + \delta]$;
2. the equation $u(x, t) = \alpha$ on the interval $x \in [\bar{a} - \delta, 1]$ has a unique root $x = a(t)$;
3. $a(t)$ is continuous;
4. the function

$$b(t) = \max_{s \in [0, t]} a(s) \tag{3.1}$$

satisfies $b(t) \in [\bar{a}, \bar{a} + \delta]$;

5. the equation $u(x, t) = \beta$ on the interval $x \in [0, b(t)]$ has no roots.

Proof. The assertions of the lemma follow from the fact that $u, u_x \in C^\gamma(\bar{Q}_T)$ for all sufficiently small T (with the norms in $C^\gamma(\bar{Q}_T)$ bounded uniformly with respect to small T), $u(x, 0) = \varphi(x)$, and $\varphi(x)$ is transverse (see Condition 2.2). \square

By possibly decreasing T and δ , we see that Lemma 3.1 holds for \hat{u} with some functions $\hat{a}(t)$ and $\hat{b}(t) = \max_{s \in [0, t]} \hat{a}(s)$ instead of $a(t)$ and $b(t)$.

Now the key observation is that the hysteresis \mathcal{H} acting on transverse functions u and \hat{u} on the time interval $[0, T]$ can be represented in terms of the free boundaries $b(t)$ and $\hat{b}(t)$ as follows:

$$\begin{aligned} \mathcal{H}(\xi_0(x), u(x, \cdot))(t) &= \begin{cases} H_1(u(x, t)), & 0 \leq x \leq b(t), \\ H_2(u(x, t)), & b(t) < x \leq 1, \end{cases} \\ \mathcal{H}(\xi_0(x), \hat{u}(x, \cdot))(t) &= \begin{cases} H_1(\hat{u}(x, t)), & 0 \leq x \leq \hat{b}(t), \\ H_2(\hat{u}(x, t)), & \hat{b}(t) < x \leq 1. \end{cases} \end{aligned} \tag{3.2}$$

The following lemma allows us to estimate the distance between the free boundaries $b(t)$ and $\hat{b}(t)$ in terms of the difference between u and \hat{u} .

Lemma 3.2. *Let T be the number from Lemma 3.1. Then*

$$\|b - \hat{b}\|_{C[0, T]} \leq \frac{1}{\bar{\varphi}} \|u - \hat{u}\|_{C(\bar{Q}_T)}.$$

Proof. It follows from the definition of $b(t)$ and $\hat{b}(t)$ that

$$\|b - \hat{b}\|_{C[0, T]} \leq \|a - \hat{a}\|_{C[0, T]}.$$

On the other hand, using Lemma 3.1, we obtain for any $t \in [0, T]$

$$|a(t) - \hat{a}(t)| \leq \frac{1}{\bar{\varphi}} |u(a(t), t) - \hat{u}(a(t), t)| \leq \frac{1}{\bar{\varphi}} \|u - \hat{u}\|_{C(\bar{Q}_T)},$$

which completes the proof. \square

Now we can prove Theorem 2.1.

1. Denote $w = u - \hat{u}$. The function w satisfies the linear parabolic equation

$$w_t = w_{xx} + h(x, t), \quad (x, t) \in Q_T, \quad (3.3)$$

where $h(x, t) = \mathcal{H}(\xi_0(x), u(x, \cdot)) - \mathcal{H}(\xi_0(x), \hat{u}(x, \cdot))$, and the zero boundary and initial conditions. Obviously, $h \in L_\infty(Q_T)$, and the function w can be represented via the Green function $G(x, y, t, s)$ of the heat equation with the Neumann boundary conditions:

$$w(x, t) = \int_0^t \int_0^1 G(x, y, t, s) h(y, s) dy ds.$$

Therefore, using the estimate

$$|G(x, y, t, s)| \leq \frac{k_1}{\sqrt{t-s}} e^{-(x-y)^2/(4(t-s))} \leq \frac{k_1}{\sqrt{t-s}}, \quad 0 < s < t,$$

with $k_1 > 0$ not depending on $(x, t) \in Q_T$ and $T > 0$ (see, e.g., [7]), we obtain

$$|w(x, t)| \leq k_1 \int_0^t \frac{ds}{\sqrt{t-s}} \int_0^1 h(y, s) dy. \quad (3.4)$$

2. Let us estimate the interior integral in (3.4) for a fixed s . We assume that $b(s) < \hat{b}(s)$ (the case $b(s) \geq \hat{b}(s)$ is treated analogously). Then, due to (3.2),

$$h(y, s) = \begin{cases} H_1(u) - H_1(\hat{u}), & 0 < y < b(s), \\ H_2(u) - H_1(\hat{u}), & b(s) < y < \hat{b}(s), \\ H_2(u) - H_2(\hat{u}), & \hat{b}(s) < y < 1. \end{cases}$$

2.1. Assertion 5 in Lemma 3.1 implies that

$$u(y, s) < \beta, \quad \hat{u}(y, s) < \beta$$

on the closed set $\{(y, s) : y \in [0, b(s)], s \in [0, T]\}$. Hence, the values $\beta - u(y, s)$ and $\beta - \hat{u}(y, s)$ are separated from 0. Therefore, using Condition 2.1, we obtain

$$\begin{aligned} \int_0^{b(s)} |h(y, s)| dy &\leq \int_0^{b(s)} \frac{M}{(\beta - u(y, s))^\sigma + (\beta - \hat{u}(y, s))^\sigma} |u(y, s) - \hat{u}(y, s)| dy \\ &\leq k_2 \int_0^{b(s)} |u(y, s) - \hat{u}(y, s)| dy \leq k_2 \|w\|_{C(\overline{Q}_T)}, \end{aligned} \quad (3.5)$$

where $k_2 > 0$ and the constants $k_3, k_4, k_5 > 0$ below do not depend on $s \in [0, T]$.

2.2. By the boundedness of $H_1(\hat{u})$ and $H_2(u)$ for $(y, s) \in \overline{Q}_T$, we have

$$\int_{b(s)}^{\hat{b}(s)} |h(y, s)| dy \leq k_3 \int_{b(s)}^{\hat{b}(s)} dy \leq k_3 \|b - \hat{b}\|_{C[0, T]}.$$

Applying Lemma 3.2 yields

$$\int_{b(s)}^{\hat{b}(s)} |h(y, s)| dy \leq \frac{k_3}{\overline{\varphi}} \|w\|_{C(\overline{Q}_T)}. \quad (3.6)$$

2.3. Let δ be the number from Lemma 3.1, and let $\hat{b}(s) < y < \overline{a} + \delta$. Then, using assertion 1 in Lemma 3.1 and the mean-value theorem, we have (see Fig. 3.1)

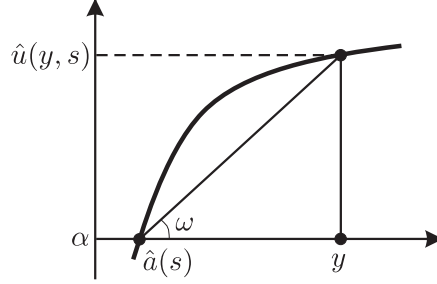


Figure 3.1: The mean-value theorem for $\hat{u}(y, s)$: $\tan \omega \geq \overline{\varphi}$

$$|\hat{u}(y, s) - \alpha| = \hat{u}(y, s) - \hat{u}(\hat{a}(s), s) \geq (y - \hat{a}(s))\overline{\varphi} \geq (y - \hat{b}(s))\overline{\varphi}.$$

Similarly,

$$|u(y, s) - \alpha| \geq (y - \hat{b}(s))\overline{\varphi}.$$

Taking into account these two inequalities and using Condition 2.1, we obtain

$$\begin{aligned} \int_{\hat{b}(s)}^{\overline{a}+\delta} |h(y, s)| dy &\leq \frac{M}{2} \int_{\hat{b}(s)}^{\overline{a}+\delta} \frac{|u(y, s) - \hat{u}(y, s)|}{(y - \hat{b}(s))^\sigma} dy \\ &\leq \frac{M\|w\|_{C(\overline{Q}_T)}}{2} \int_{\hat{b}(s)}^{\overline{a}+\delta} \frac{1}{(y - \hat{b}(s))^\sigma} dy \leq k_4 \|w\|_{C(\overline{Q}_T)}. \end{aligned} \quad (3.7)$$

2.4. Finally, assertions 1 and 2 in Lemma 3.1 imply that

$$u(y, s) > \alpha, \quad \hat{u}(y, s) > \alpha$$

on the closed set $[\overline{a} + \delta, 1] \times [0, T]$. Hence, the values $u(y, s) - \alpha$ and $\hat{u}(y, s) - \alpha$ are separated from 0. Therefore, due to Condition 2.1,

$$\begin{aligned} \int_{\overline{a}+\delta}^1 |h(y, s)| dy &\leq \int_{\overline{a}+\delta}^1 \frac{M}{(u(y, s) - \alpha)^\sigma + (\hat{u}(y, s) - \alpha)^\sigma} |u(y, s) - \hat{u}(y, s)| dy \\ &\leq k_5 \int_{\overline{a}+\delta}^1 |u(y, s) - \hat{u}(y, s)| dy \leq k_5 \|w\|_{C(\overline{Q}_T)}. \end{aligned} \quad (3.8)$$

3. Combining estimate (3.4) with inequalities (3.5)–(3.8), we obtain

$$|w(x, t)| \leq k_6 \|w\|_{C(\overline{Q}_T)} \int_0^t \frac{ds}{\sqrt{t-s}} = 2k_6 T^{1/2} \|w\|_{C(\overline{Q}_T)}.$$

Since k_6 does not depend on x and t , the latter inequality is equivalent to $\|w\|_{C(\overline{Q}_T)} \leq 2k_6 T^{1/2} \|w\|_{C(\overline{Q}_T)}$. Since k_6 does not depend on (small) T either, it follows that $w = 0$ provided that $T > 0$ is small enough. \square

A Connection with slow-fast systems

It is known [4, 10] that hysteresis may approximate solutions of ordinary differential equations with a small parameter $\varepsilon > 0$ and a “bistable” right-hand side. Combining such an ordinary differential equation with the reaction-diffusion equation (2.4), one obtains a slow-fast system of the form

$$\begin{cases} u_t = u_{xx} + f(u, v), \\ \varepsilon v_t = g(u, v), \end{cases} \quad (\text{A.1})$$

where $\varepsilon > 0$, $f(u, v)$ and $g(u, v)$ are smooth functions of class C^∞ , and $g(u, v)$ satisfies the following condition (see Fig. A.1).

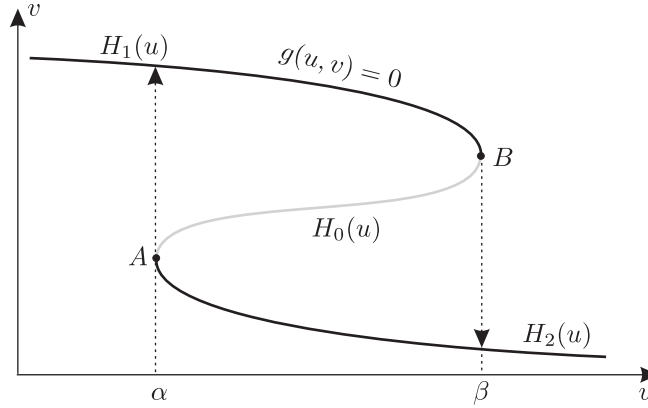


Figure A.1: The nullcline of $g(u, v)$

Condition A.1. 1. *There are two numbers $\alpha < \beta$ such that the equation $g(u, v) = 0$ has a unique root $v = H_1(u)$ for $u < \alpha$ and $v = H_2(u)$ for $u > \beta$; three distinct roots $v = H_1(u)$, $v = H_2(u)$, and $v = H_0(u)$ for $u \in (\alpha, \beta)$; two distinct roots $v = H_1(\alpha)$ and $v = H_2(\alpha) = H_0(\alpha)$; two distinct roots $v = H_1(\beta) = H_0(\beta)$ and $v = H_2(\beta)$.*

2. *the functions $H_1(u)$ and $H_2(u)$ are locally Lipschitz continuous for $u < \beta$ and $u > \alpha$, respectively.*

3. *Let $A = (\alpha, H_2(\alpha))$ and $B = (\beta, H_1(\beta))$. Then $\frac{\partial g(A)}{\partial u} \neq 0$ and $\frac{\partial g(B)}{\partial u} \neq 0$.*

4. *$\frac{\partial g(A)}{\partial v} = \dots = \frac{\partial^{n-1} g(A)}{\partial v^{n-1}} = 0$, $\frac{\partial^n g(A)}{\partial v^n} \neq 0$ for some even $n \geq 2$ and the same holds at the point B .*

5. *$g(u, v) > 0$ (< 0) if a point (u, v) lies to the left (right) from the curve $g(u, v) = 0$ on the plane (u, v) .*

The simplest example of such a bistable nonlinearity typical, e.g., for the FitzHugh–Nagumo systems is given by

$$g(u, v) = u + v - v^3.$$

As we have mentioned before, the natural conjecture is that the solutions of the slow-fast system (A.1) approximate as $\varepsilon \rightarrow 0$ the solution of Eq. (2.4) with hysteresis \mathcal{H} defined by the curves $H_1(u)$ and $H_2(u)$ from Condition A.1. We refer to [13], where singular limit analysis of traveling waves in bistable systems is done, and to [2, 14], where equations of the form $u_t = \Delta \Phi(u)$ with a cubic nonlinearity Φ are studied.

The proof of the above conjecture in the general situation is an open question, which is beyond the scope of the present paper. However, it is clear that the proof would deal with hysteresis defined by non-Lipschitz curves. The following result assures that the non-Lipschitz curves $H_1(u)$ and $H_2(u)$ from Condition A.1 satisfy Condition 2.1 and thus fit into our theory.

Lemma A.1. *Let $g(u, v)$ satisfy Condition A.1. Then the functions $H_1(u)$ and $H_2(u)$ from Condition A.1 satisfy Condition 2.1 with $\sigma = (n - 1)/n$.*

Proof. 1. Without loss of generality, we assume that $A = 0$ and prove the lemma for the function $H_2(u)$. Due to item 2 in Condition A.1, it suffices to prove inequality (2.2) for u in a small neighborhood of $\alpha = 0$.

Since $\frac{\partial g(A)}{\partial u} \neq 0$, it follows from the implicit function theorem that there is a unique function $G_2(v)$ defined in a small neighborhood of 0 such that $g(G_2(v), v) \equiv 0$ and

$$G_2(0) = G_2'(0) = \dots = G_2^{(n-1)}(0) = 0, \quad G_2^{(n)}(0) > 0. \quad (\text{A.2})$$

Obviously, $G_2(v)$ and $H_2(u)$ are inverse to each other for all negative v and positive u close to 0. It is convenient to introduce the function $G(w) = G_2(-w)$ defined for small positive w . Due to (A.2), it satisfies

$$G(0) = G'(0) = \dots = G^{(n-1)}(0) = 0, \quad G^{(n)}(0) > 0 \quad (\text{A.3})$$

since n is even. Now if we denote $w = -H_2(u)$ (> 0), then $u = G_2(H_2(u)) = G_2(-w) = G(w)$. Therefore, the inequality for H_2 in Condition (2.1), which we have to prove, is equivalent to the following:

$$\frac{G(w) - G(\hat{w})}{w - \hat{w}} \geq M \left([G(w)]^{\frac{n-1}{n}} + [G(\hat{w})]^{\frac{n-1}{n}} \right) \quad (\text{A.4})$$

for all $0 < \hat{w} < w \leq \varepsilon_0$, where $\varepsilon_0 > 0$ is sufficiently small and $M > 0$ does not depend on w, \hat{w} .

2. Expanding $G(w)$ by the Taylor formula about $w = \hat{w}$, we have

$$\frac{G(w) - G(\hat{w})}{w - \hat{w}} = G'(\hat{w}) + \sum_{k=2}^{n-1} \frac{G^{(k)}(\hat{w})}{k!} (w - \hat{w})^{k-1} + \frac{G^{(n)}(\xi)}{n!} (w - \hat{w})^{n-1}, \quad (\text{A.5})$$

where $0 \leq \xi = \xi(w, \hat{w}) \leq \varepsilon_0$.

Further, we expand $G^{(k)}(\hat{w})$, $k = 1, \dots, n-1$ by the Taylor formula about the origin, using (A.3):

$$G^{(k)}(\hat{w}) = \frac{G^{(n)}(\xi_k)}{(n-k)!} \hat{w}^{n-k},$$

where $0 \leq \xi_k = \xi_k(\hat{w}) \leq \varepsilon_0$. Substituting these expressions into (A.5) yields

$$\begin{aligned} \frac{G(w) - G(\hat{w})}{w - \hat{w}} &\geq \frac{G^{(n)}(\xi_1)}{(n-1)!} \hat{w}^{n-1} + \frac{G^{(n)}(\xi)}{n!} (w - \hat{w})^{n-1} \\ &\geq \frac{G^{(n)}(0)}{2(n-1)!} \left(\hat{w}^{n-1} + \frac{1}{n} (w - \hat{w})^{n-1} \right), \end{aligned} \quad (\text{A.6})$$

where we assume ε_0 so small that $G^{(n)}(\xi_k) \geq \frac{G^{(n)}(0)}{2}$ and $G^{(n)}(\xi) \geq \frac{G^{(n)}(0)}{2}$.

It is easy to show that

$$\hat{w}^{n-1} + \frac{1}{n} (w - \hat{w})^{n-1} \geq M_1 (w^{n-1} + \hat{w}^{n-1})$$

for some $M_1 > 0$ not depending on w and \hat{w} , $0 < \hat{w} < w$. Hence, inequality (A.6) implies

$$\frac{G(w) - G(\hat{w})}{w - \hat{w}} \geq M_2 (w^{n-1} + \hat{w}^{n-1}), \quad (\text{A.7})$$

where $M_2 > 0$ does not depend on w and \hat{w} , $0 < \hat{w} < w \leq \varepsilon$. Combining (A.7) and the relations

$$G(w) = \frac{G^{(n)}(\zeta)}{n!} w^n \leq \frac{2G^{(n)}(0)}{n!} w^n$$

with $0 \leq \zeta = \zeta(w) \leq \varepsilon_0$ yields (A.4). \square

Acknowledgement: The authors are grateful to Willi Jäger for drawing their attention to the field of hysteresis. The research of the first author was supported by the DFG project SFB 910, by the DAAD program G-RISC, and by the RFBR (project 10-01-00395-a). The research of the second author was supported by the Alexander von Humboldt Foundation.

References

- [1] H. W. Alt, *On the thermostat problem*. Control Cyb., **14**, 171–193 (1985).
- [2] L. C. Evans, M. Portilheiro, *Irreversibility and hysteresis for a forward-backward diffusion equation*. Math. Models Methods Appl. Sci., **14**, no. 11, 1599–1620 (2004).
- [3] P. Gurevich, R. Shamin, S. Tikhomirov, *Reaction-diffusion equations with spatially distributed hysteresis*. Preprint.
- [4] P. Krejčí, *The hysteresis limit in relaxation oscillation problems*. Journal of Physics: Conference Series, **22**, 103–123 (2005).
- [5] F. C. Hoppensteadt, W. Jäger, *Pattern formation by bacteria* Lecture Notes in Biomathematics **38**, 68–81 (1980).
- [6] F. C. Hoppensteadt, W. Jäger, C. Poppe, *A hysteresis model for bacterial growth patterns*. Modelling of Patterns in Space and Time, Lecture Notes in Biomath. **55** (Springer), 123–134 (1984).
- [7] S. D. Ivashiĭn *Green’s matrices of boundary value problems for Petrovski parabolic systems of general form, II*, Mat. Sb., **114(156)**, No. 4, 523–565 (1981); English transl. in Math. USSR Sbornik, **42** 461–489 (1982).
- [8] A. M. Il’in, B. A. Markov, *Nonlinear diffusion equation and Liesegang rings*. Doklady Akademii Nauk, **440**, No. 2, 164–167 (2011); English translation: Doklady Mathematics, **84**, No. 2, 730–733 (2011).
- [9] J. Kopfova, *Hysteresis in biological models*. Journal of Physics: Conference Series, **55**, 130–134 (2006).
- [10] E. F. Mishchenko, N. Kh. Rozov, *Differential equations with small parameters and relaxation oscillations*, Plenum (1980).
- [11] M. A. Krasnosel’skii, A. V. Pokrovskii. *Systems with Hysteresis*. Springer-Verlag. Berlin–Heidelberg–New York (1989). Translated from Russian: *Sistemy s Gisterezisom*. Nauka. Moscow (1983).
- [12] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Uraltseva. *Linear and Quasilinear Equations of Parabolic Type*. Nauka, Moscow, 1967; English translation: Amer. Math. Soc., Providence, RI, 1968.
- [13] Y. Nishiura, M. Mimura, H. Ikeda, H. Fujii. *Singular limit analysis of stability of traveling wave solutions in bistable reaction-diffusion systems*, SIAM J. Math. Anal., **21**, No. 1, 85–122 (1990).
- [14] P. I. Plotnikov, *Passing to the limit with respect to the viscosity in an equation with variable parabolicity direction*, Differential Equations, **30**, 614–622 (1994).
- [15] A. Visintin *Differential Models of Hysteresis*. Springer-Verlag. Berlin — Heidelberg (1994).
- [16] A. Visintin *Evolution problems with hysteresis in the source term*. SIAM J. Math. Anal., **17**, 1113–1138 (1986).